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Cite as: Physics of Plasmas **3**, 1916 (1996); <https://doi.org/10.1063/1.871987>

Submitted: 07 November 1995 . Accepted: 05 December 1995 . Published Online: 04 June 1998

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Anomalous pinch effect and energy exchange in tokamaks*

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(Received 7 November 1995; accepted 5 December 1995)

It is shown that, under very generic assumptions of the nature of plasma turbulence in an inhomogeneous magnetic field, the anomalous particle and energy fluxes, in addition to usual diffusive terms, involve purely convective terms not directly associated with density or temperature gradients. The anomalous convective transport results from the conservation, on turbulent time scale, of adiabatic invariants of particle motion in the inhomogeneous magnetic field and the Liouville-theorem constraint on the microscopic dynamics, thus furnishing the mechanism of the anomalous pinch effect in tokamaks. This theory also predicts an electron-turbulence energy exchange, which can be interpreted as a turbulent enhancement of the electron-ion energy exchange. Collisions introduce important modifications to the turbulent mechanisms of the pinch effect. It is argued that the nondiffusive effects are intrinsic to tokamak transport and should be included in power-balance analyses. © 1996 American Institute of Physics. [S1070-664X(96)90505-4]

I. INTRODUCTION

The tokamak energy confinement, clearly one of the most important reactor characteristics, is routinely measured by the heat diffusivity χ relating the thermal flux \mathbf{q} to the temperature gradient ∇T as $\mathbf{q} = -(3n/2)\chi\nabla T$. It is not as clear, however, whether the transport in tokamaks is of pure diffusive nature and whether χ is the most relevant transport characteristic. Indeed, there exists a body of experimental evidence that the heat flux \mathbf{q} involves a significant inward convective component¹⁻⁵ and even bears certain nonlocal features⁶⁻⁸ not easily reducible to combined diffusive and convective fluxes.

The phenomenon of profile consistency, or resilience⁹ observed for plasma temperature,^{3,10,11} density,^{1,12,13} or pressure⁴ has been known for a long time, yet the underlying particle and energy pinch effects have been much less studied than the particle and thermal diffusivities. For the one thing, this has been due to the lack of a clear physical explanation of the turbulent, or anomalous pinch effect. Also, the experimental measurements of off-diagonal terms of the transport matrix and the convective fluxes are much more complicated, especially for the energy transport, where the pinch effect is often obscured by a substantial energy source at the plasma center. Still, the typically peaked tokamak density profiles in the absence of significant particle sources at the center indicate a significant particle pinch.¹⁴

In our terminology, the pinch effect stands for particle and energy fluxes not directly associated with either density or temperature gradients. The usual off-diagonal (cross) fluxes appear in the neoclassical theory (cf. Ref. 15), which also predicts an inward pinch velocity proportional to the toroidal loop voltage.^{16,17} This neoclassical pinch effect is typically too weak to explain the experiment, but it can be made stronger by phenomenologically introducing an enhanced electron-electron collision frequency.¹⁸⁻²⁰

Turbulent mechanisms of the pinch effect are not neces-

sarily related to the inductive loop voltage. Several quasilinear theories reported various pinch terms in the anomalous fluxes.²¹⁻²⁴ The pinch terms were also present in the bounce-average equations of Ref. 25, although dropped in the final results.

Smolyakov *et al.*²⁶ highlighted the role of the Liouville theorem which precludes quasilinear convective fluxes in the canonical phase space (\mathbf{x}, \mathbf{v}) , and pointed out that such fluxes can appear upon the projection onto the coordinate space \mathbf{x} . More recently, an important clarification of the physics of the anomalous pinch effect in tokamaks was made by Yankov^{27,28} and Nycander and Yankov,^{29,30} who proposed that turbulent plasma in the absence of particle and energy sources assumes a state close to the so-called *turbulent equipartition*. This state corresponds to a phase-space density uniform on the surfaces of the two constant adiabatic invariants $\mu = mv_{\perp}^2/(2B)$ and $J = \oint mv_{\parallel} dl_{\parallel}$, a version of the ergodic hypothesis where the adiabatic invariants play the role of the no longer conserved energy. The applicability of the statistical mechanics to self-consistent plasma turbulence is not clear, but some predictions of this approach are in a qualitative agreement with experiments. In the state of turbulent equipartition, the plasma density and temperature are inhomogeneous, and there are no particle or energy fluxes. Such a state is usually marginally stable to the modes potentially responsible for driving the plasma to this state.^{28,30-32} When applied to collisionless toroidal plasma with a large aspect ratio $R/r \gg 1$, the turbulent equipartition corresponds to the distribution function

$$f(\mathbf{x}, J, \mu) = f_0(J, \mu)/q(r), \quad (1)$$

where $f(\mathbf{x}, J, \mu)$ is the particle density with the given adiabatic invariants, $q(r) = rB_{\phi}/(RB_{\theta})$ is the safety factor, B_{ϕ} and B_{θ} are the toroidal and the poloidal magnetic fields, and r and R are the minor and the major radii, respectively. Here and in what follows, we adopt a uniform convention whereby f denotes the particle density in the phase space specified by f 's arguments (except time). The tokamak safety factor profile, $q(r)$, is typically an increasing function of the minor radius, and the density (1) is peaked toward the center,

*Paper 21B1, Bull. Am. Phys. Soc. **40**, 1668 (1995).

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thus constituting the collisionless pinch effect of Yankov. Equation (1) describes the distribution of trapped particles subject to radial drifts in low-frequency electrostatic modes, whereas passing particles effectively average out the effect of turbulence and eventually come to a collisional equilibrium with the inhomogeneously distributed trapped particles.

The conjecture^{27,28} that the profile of the total density $n(r)$ will essentially follow the turbulent equipartition (1), namely $n(r)q(r)=\text{const}$, was questioned in the previous letter,³³ where collisions were taken into account and the peakedness of the plasma density, in a thin torus with $r/R \leq 1$, was found to be $O(r/R)$. The reason lies in a cancellation characteristic of the thin torus limit and, in the simplest terms, is as follows. The integration of Eq. (1) over $J \propto v_{\parallel}L$ involves the factor of the connection length $L=qR$, and the strong radial dependence via $q(r)$ drops out of the particle density $n = \int f(\mathbf{x}, J, \mu) dJ d\mu$. Put differently, an isotropic Maxwellian with constant density and temperature rewrites in the (\mathbf{x}, J, μ) coordinates exactly as Eq. (1) in the trapped region for $r/R \leq 1$. It is not even *a priori* evident that taking into account higher-order corrections will result in an inward pinch. Under some simplifying assumptions made in Ref. 33, the convection was found inward for $d \log q/d \log r > -3/8$.

In this paper, without restriction to particular turbulent modes, we discuss the physics of the anomalous pinch effect in a magnetically axisymmetric tokamak. The microscopic dynamics of particles in the turbulent fields is Hamiltonian, which imposes certain constraints on the dynamics even without knowing the details of the turbulence. The most fundamental constraint is the Liouville theorem, which predicts a pure particle diffusion in the action space upon introducing suitable action-angle variables. Under additional constraints, such as the conservation of the first adiabatic invariant μ (for turbulent frequency ω much less than the gyro frequency ω_c) and the second invariant J (for ω much less than the parallel bounce frequency ω_b), the pure diffusion along the third action translates into both a diffusivity and an average pinch velocity in the coordinate space. The pinch velocity arises as a result of the magnetic field inhomogeneity, as if the gradient of \mathbf{B} were a ‘‘thermodynamic force,’’ due to the coordinate-dependent constraints of the adiabatic invariance. The formalism of the action-space diffusion predicts collisionless turbulent equipartitions in agreement with Refs. 27 and 29 and also allows to include collisions in the transport analysis.

We address both particle and energy transport, with a specific emphasis on electrons for which the theory’s small parameters, such as ω/ω_{be} , are better justified and for which more anomalies are observed experimentally. The source of plasma turbulence is not discussed, and the transport is studied for the given, under fairly generic assumptions, electrostatic fluctuations. Of course, the test-particle approach lacks the self-consistent dynamics of turbulence; therefore, only those transport features, which are not too sensitive to the assumptions of the turbulence spectrum, are valuable predictions of this theory. The anomalous particle and energy pinch effects and the anomalous energy exchange are among these features.

The paper is organized as follows. In Sec. II the role of canonical coordinates is discussed in the context of turbulent diffusion. In Sec. III toroidal magnetic flux coordinates are used to represent the bounce averaged turbulent drifts of trapped electrons in a Hamiltonian form and to formulate the kinetic response of collisional electrons to a low-frequency electrostatic turbulence. In Sec. IV the kinetic equation is solved using the large parameter $\nu_e \tau_E \gg 1$, where ν_e is the electron collision frequency and τ_E is the confinement time, and electron transport equations are obtained for the regime $\omega_{be} \gg \omega \gg \nu_e$. The ‘‘canonical’’ profiles corresponding to the absence of particle and energy fluxes are discussed in Sec. V. In Sec. VI we discuss implications and possible extensions of our results. Several auxiliary calculations are organized in appendices. In Appendix A, we summarize the large-aspect-ratio tokamak limit of the arbitrary toroidal geometry used throughout most of the paper. Here we also consider the regime of very low-frequency modes, $\omega_{be} \gg \nu_e \gg \omega$, in which the pinch effect is also $O(r/R)$. In Appendix B, the transport of passing particles by low-frequency turbulence is studied.

II. THE ROLE OF CANONICAL COORDINATES IN TURBULENT DIFFUSION

Turbulent transport is caused by the random walk of particles in randomly fluctuating fields. Pure diffusion is an unbiased random walk of a particle with no average displacement, $\langle x \rangle = 0$, and with a linearly growing mean square displacement $\langle x^2 \rangle = 2Dt$, D being the diffusivity. The intuitive motivation behind the diffusion approximation is that zero-average fluctuating fields imply a zero-average velocity. However, the well-known example of the ponderomotive force (cf. Ref. 34) says that the zero-average, at a given \mathbf{x} , acceleration $\ddot{\mathbf{x}} = \mathbf{a}(\mathbf{x}, t)$ implies an average force, if the envelope of the HF field \mathbf{a} is inhomogeneous. Even more transparently, the pure-diffusion approximation fails for the first-order (drift-type) equation $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t)$ with the zero-average, at fixed \mathbf{x} , velocity \mathbf{v} , because the pure diffusion cannot be independent of the coordinates used to track down the particle position. Indeed, if, in some coordinate x , the particle is executing an unbiased random walk with the diffusivity D , in another coordinate, say $x' = x^2$, the particle has a nonzero average velocity: $\langle x' \rangle = 2Dt$.

It is the transition from single-particle dynamics to an average Fokker-Planck description, where the average (pinch) velocity does or does not appear. Below we show that there is no average velocity in the canonical action variables, but, before doing so, consider what happens to a diffusion equation when variables are changed. Suppose there is no average particle velocity in some coordinates \mathbf{x} : $\partial_t n = \partial_{\mathbf{x}} \cdot (\mathbf{D} \cdot \partial_{\mathbf{x}} n)$, where n is the particle density in the \mathbf{x} space. (In order to have the terms in the Fokker-Planck equation simply related to the particle motion, the density must be thus defined.) Upon the change of the variables, $\mathbf{x} \rightarrow \mathbf{x}'(\mathbf{x})$, and introducing the \mathbf{x}' -density $n'(\mathbf{x}', t) = n(\mathbf{x}, t) |\partial(\mathbf{x})/\partial(\mathbf{x}')|$, we obtain the transport equation with a convection term:

$$\partial_t n' = \partial_{\mathbf{x}'} \cdot (\mathbf{D}' \cdot \partial_{\mathbf{x}'} n' - \mathbf{V}' n'), \quad D'^{ij} = D^{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l},$$

$$\mathbf{V}' = \mathbf{D}' \cdot \partial_{\mathbf{x}'} \log \left| \frac{\partial(\mathbf{x})}{\partial(\mathbf{x}')} \right|. \quad (2)$$

Thus, a nonlinear change of coordinates renders pure diffusion biased; that is, with an average velocity in the new coordinates. Obversely, an existing bias can be removed by changing to another, “natural” set of coordinates in which there is no convective particle flux. Generally speaking, how do we know that Cartesian coordinates, or the tokamak minor radius r , are such natural coordinates for the purpose of anomalous plasma transport? In fact, the observed pinch effect suggests that Cartesian spatial coordinates are not, and below we show that the poloidal magnetic flux $\psi(r)$ is the natural radial coordinate in which, without collisions, trapped electrons diffuse purely and tend to distribute homogeneously, with the per-unit-flux density $N_e(\psi) \propto n(r)q(r) = \text{const}$.

In general, if the particle motion is Hamiltonian and stochastic, the canonical coordinates $\mathbf{z} = (\mathbf{p}, \mathbf{q})$ furnish the natural coordinates in the above sense. This is due to the Liouville theorem which says that the phase-space velocity, $\dot{\mathbf{z}} = \mathbf{u}(\mathbf{z}, t)$, is incompressible, $\partial_{\mathbf{z}} \cdot \mathbf{u} = 0$, and therefore the continuity equation for the distribution function $f(\mathbf{z}, t)$, $\partial_t f + \partial_{\mathbf{z}} \cdot (f\mathbf{u}) = 0$, has the exact phase-space-uniform solution $f(\mathbf{z}, t) = \text{const}$. It means that if the Fokker-Planck equation,

$$\partial_t F(\mathbf{z}, t) = \partial_{\mathbf{z}} \cdot [\mathbf{D}(\mathbf{z}) \cdot \partial_{\mathbf{z}} F - \mathbf{V}(\mathbf{z}) F], \quad (3)$$

for the coarse-grained (averaged) distribution function $F(\mathbf{z}, t)$, is a valid representation of the turbulent transport, it must also have a spatially uniform solution, and hence the compressible part of the average velocity \mathbf{V} must be zero.

This line of reasoning has long been known in the physics of the Earth’s magnetosphere and Van Allen radiation belts,^{35–38} where canonical action-angle variables form the natural basis for the description of turbulent diffusion. In the fusion literature, the quasi-linear diffusion in the action space was also studied,^{39–41} but not connected with the pinch effect. A notable exception is Refs. 42 and 32, where a fusion reactor based on a dipole magnetic field was proposed with a specific emphasis on the strongly inhomogeneous density profile, $n(\mathbf{x}) \propto R^{-4}$, and the direct analogy with magnetosphere was pointed out. The same analogy motivated the “negative heat conductivity” suggested by Kadomtsev.⁴³

The standard drift representation of the particle motion in the Earth’s magnetic field is in terms of the action-angle variables $\mathbf{z} = (\mu, \alpha_\mu, J, \alpha_J, \psi, \alpha_\psi)$, where $\mu = mv_\perp^2/(2B)$ is the magnetic moment (the action of the gyro motion), α_μ is the gyro angle, $J = \oint mv_\parallel dl_\parallel$ is the parallel bounce action, α_J is the corresponding bounce angle, ψ is the (poloidal) magnetic flux through the closed drift surface swept by the bouncing particle, and α_ψ is the corresponding (toroidal) angle. Without external perturbations, the actions μ , J , and ψ are constant, and the corresponding angles are linearly changing with the cyclic frequencies $\omega_c \gg \omega_b \gg \omega_\phi$ (the cyclotron, the longitudinal bounce, and the toroidal orbit precession frequency, respectively). If an external perturbation to the particle Hamiltonian is present, some of the actions are no longer conserved, depending on the characteristic frequency of the perturbation. Only those actions, whose corre-

sponding angular velocities are much greater than the perturbation frequency ω , remain conserved as adiabatic invariants.⁴⁴ For example, if $\omega_c \gg \omega_b \gg \omega \gg \omega_\phi$, the first two adiabatic invariants, μ and J , are conserved, and the third invariant ψ is broken thus introducing radial diffusion of bouncing particles. If we are not interested in the angular coordinates α_i , the distribution function $f(\mathbf{I}, \boldsymbol{\alpha})$ (where $I^{1,2,3} = \mu, J, \psi$) can be integrated over the angles $\boldsymbol{\alpha}$, and Eq. (3) becomes the action-diffusion equation:

$$\partial_t f(\mathbf{I}, t) = \partial_{I^i} [D^{ij}(\mathbf{I}) \partial_{I^j} f(\mathbf{I}, t)]. \quad (4)$$

The conservation of $I^1 = \mu$ and, possibly, $I^2 = J$ means that the components of the diffusion tensor D^{ij} involving the corresponding indices are all zero.

Alternatively, for $\omega_b \gg \omega$, the single-particle motion can be averaged over the fast gyration and bouncing before introducing the kinetic description. The bounce-average drift of the guiding center can be written in the Hamiltonian form using the Clebsch coordinates α_ψ and ψ defined locally for a general magnetic field $\mathbf{B}(\mathbf{x})$ by

$$\mathbf{B} = \nabla \alpha_\psi \times \nabla \psi / (2\pi). \quad (5)$$

The magnetic field lines are given by the intersection of the surfaces $\psi = \text{const}$ and $\alpha_\psi = \text{const}$. Then the equations of average particle motion across the magnetic field lines take the canonical Hamiltonian form:^{44–46}

$$\dot{\psi} = (2\pi c/e) \partial_{\alpha_\psi} H, \quad \dot{\alpha}_\psi = -(2\pi c/e) \partial_\psi H. \quad (6)$$

The Hamiltonian $H = H(\mu, J, \psi, \alpha_\psi, t) = \langle e\phi(\mathbf{x}, t) + mv_\parallel^2/2 + \mu B(\mathbf{x}) \rangle$ is the bounce-average particle energy expressed via the adiabatic invariants μ and J as parameters, c is the speed of light, e is the charge, and $\phi(\mathbf{x}, t)$ is the electrostatic potential.

In a two dimensional geometry with $B = B_z(x, y)$ and $\phi = \phi(x, y, t)$, the guiding-center diffusion in the (ψ, α_ψ) plane leads to the uniform Clebsch-space density $f(\psi, \alpha_\psi) = \text{const}$, corresponding to the volume density $n \equiv f(x, y) = f(\psi, \alpha_\psi) |\partial(\psi, \alpha_\psi) / \partial(x, y)| \propto B(x, y)$. A numerical test of this turbulent equipartition was presented in Ref. 47. In this geometry, collisions do not alter the final density profile, although they do affect the temperature profile.^{28,48}

There is a straightforward analogy between this two dimensional pinch effect and the “dynamic friction”⁴⁹ for the stochastic field lines in the magnetic field $\mathbf{B} = B(x, y) \hat{\mathbf{z}} + \nabla \delta A(x, y, z) \times \hat{\mathbf{z}}$, where δA is a small perturbation. The equation of a magnetic field line is equivalent to the $\mathbf{E} \times \mathbf{B}$ drift equation, where δA plays the role of ϕ and z stands for time. The particle density n in the $\mathbf{E} \times \mathbf{B}$ problem then translates into the density of magnetic field lines, which is clearly proportional to $B(x, y)$.

III. ELECTRON DRIFT KINETICS IN TOROIDAL GEOMETRY

The action-angle variables described in Sec. II equally apply to an arbitrary magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ possessing flux surfaces $\psi = \text{const}$:

$$\mathbf{B} = \frac{\nabla \chi \times \nabla \theta - \nabla \psi \times \nabla \varphi}{2\pi} = \frac{\nabla[\varphi - q(\psi)\theta] \times \nabla \psi}{2\pi}. \quad (7)$$

Here ψ is the poloidal magnetic flux between the surface and the magnetic axis, $\chi(\psi)$ is the toroidal flux within the surface, $q = d\chi/d\psi$ is the safety factor, θ and φ are respectively the poloidal and the toroidal 2π -periodic angle coordinates. (Do not confuse the toroidal angle φ with the electrostatic potential denoted by ϕ .) In the case of axisymmetry, $\partial_\varphi = 0$, φ is simply the azimuthal angle. The Jacobian $\sqrt{g} = |\nabla\psi \times \nabla\theta \cdot \nabla\varphi|^{-1} = qR/(2\pi B_\varphi)$ of the magnetic flux coordinates⁵⁰ (ψ, θ, φ) defines the connection length $L(\psi, \theta) = 2\pi B \sqrt{g} = qRB/B_\varphi$ entering the magnetic field line length element $d\ell_{\parallel} = Ld\theta$. Here $R(\psi, \theta)$ is the local major radius and $\mathbf{B}_\varphi(\psi, \theta) = (q/2\pi)\nabla\psi \times \nabla\theta$ the toroidal magnetic field.

For low-frequency potential fluctuations, $\omega \ll \omega_b$, the parallel action

$$J = \oint mv_{\parallel} d\ell_{\parallel} = \int_{-\pi}^{\pi} Ld\theta \Theta(\epsilon - \mu B - e\phi) \times \sqrt{(m/2)(\epsilon - \mu B - e\phi)}, \quad (8)$$

is adiabatically conserved for trapped particles but introduced, as a phase-space variable, for both trapped and passing particles alike with the help of the step function Θ . Here ϵ is total particle energy and the integration in Eq. (8) is along the magnetic field line $\alpha_\psi = \text{const}$, $\psi = \text{const}$. The bounce angle $\alpha_j(\ell_{\parallel})$ is defined by the standard condition $\partial(J, \alpha_j)/\partial(mv_{\parallel}, \ell_{\parallel}) = 2\pi$, and the toroidal precession angle $\alpha_\psi = \varphi - q\theta$. If the potential fluctuations are small in amplitude, $e\phi \ll T$, the fluctuating part of the potential $\phi = \phi_0(\psi) + \tilde{\phi}(\psi, \theta, \varphi, t)$ can be neglected in the definition of J which then becomes axisymmetric. In what follows, we will need some magnetic geometry characteristics associated with J . The bounce-invariant pitch-angle variable can be introduced as either $j = J(m\mu)^{-1/2}$ or $\mathcal{B} = (\epsilon - e\phi)/\mu$, the latter one having the meaning of the maximum accessible magnetic field for the particle. The two variables are functions of one another :

$$j(\mathcal{B}, \psi) = \int_{-\pi}^{\pi} L(\psi, \theta) d\theta \Theta[\mathcal{B} - B(\psi, \theta)] \sqrt{(\mathcal{B} - B)/2}. \quad (9)$$

The dependence of $\mathcal{B}(j, \psi)$ is completely defined by the magnetic geometry. Explicit formulae for a torus with $r/R \ll 1$ are written in Appendix A. For given μ and ψ , particles with $J < J_c(\mu, \psi) = (m\mu)^{1/2} j_c(\psi)$ are trapped, and those with $J > J_c$ are passing.

Equation (7) implies that $\alpha_\psi = \varphi - q\theta$, the toroidal angle of the field line intersection with the mid-plane $\theta = 0$, and ψ , the poloidal magnetic flux, are the Clebsch variables of the magnetic field in an axisymmetric tokamak. Therefore, the bounce-average motion of a trapped particle is described by Eq. (6). The bounce averaging is appropriate if the frequency of the fluctuating fields is much less than the bounce frequency. For a generic drift-wave or ion-temperature-gradient turbulence, the frequency is of order $\omega = k_\theta cT/(eBL_n)$, where L_n is the radial scale of inhomogeneity. The ratio of this frequency to the electron bounce frequency $\omega_{be} \approx (r/R)^{1/2} v_{Te}/(qR)$ is

$$\frac{\omega}{\omega_{be}} \approx \left(\frac{m_e}{m_i}\right)^{1/2} \frac{R}{L_n} \left(\frac{R}{r}\right)^{1/2} \frac{qR}{L_n} k_\theta \rho_i, \quad (10)$$

which can vary from 1/3 to 1/50. For ions, $\omega/\omega_{bi} \gtrsim 1$, and bounce averaging makes no sense.

Using the inequalities $\omega_{be} \gg \omega, \nu_e$, where ν_e is the collision frequency, we can write the following bounce-average kinetic equation for the electrons:

$$\partial_t f(\mu, J, \psi, \alpha_\psi, t) = (2\pi c/e)[H(\mu, J, \psi, \alpha_\psi, t), f] + C(f). \quad (11)$$

Here $[a, b] \equiv \partial(a, b)/\partial(\psi, \alpha_\psi)$ is the Poisson bracket and H is the bounce-average Hamiltonian including the fluctuation potential. The bounce averaging for a passing particle on an irrational magnetic surface $q \neq m/n$ is essentially a flux-surface averaging. Thus neither H nor f depend on α_ψ in the passing region $J > J_c$, and the fluctuation transport term in Eq. (11) is zero. The turbulent transport of passing particles appears in the next order in $\omega/\omega_{be} \ll 1$ (see Appendix B).

The bounce-average collision operator can be written in the particle-conserving form

$$C[f(\mu, J, \psi, \alpha_\psi)] = -\partial_i \Gamma^i, \quad i = (\mu, J, \psi, \alpha_\psi), \quad \Gamma^i = -D^{ij} \partial_j f + U^i f, \quad (12)$$

where the diffusion D^{ij} and the dynamic friction U^i are linear integral operators on f .^{51,52} The collisional fluxes Γ^μ and Γ^J in the ‘‘velocity directions’’ drive the distribution function to the flux-surface-local Maxwellian,

$$f_0(\mu, J, \psi) = \frac{n(\psi)}{(2\pi m)^{1/2} T^{3/2}(\psi)} \exp\left[-\frac{\mu \mathcal{B}(j, \psi)}{T(\psi)}\right], \quad (13)$$

on the collisional time scale $\tau_e = \nu_e^{-1}$. The density n and the temperature T are constant on the flux surfaces $\psi = \text{const}$ because the distribution function must be constant along the particle orbits shorter than the mean-free path.

On the other hand, the radial flux Γ^ψ specifies the neoclassical transport, which relaxes the profiles of density and temperature on a much longer, neoclassical confinement time scale. According to experimental data (cf. the review Ref. 14), electron transport is at least by an order of magnitude greater than predicted by the neoclassical theory. Therefore, one can neglect the neoclassical radial flux Γ^ψ in the collision operator (12) in comparison with the Hamiltonian term in (11).

If, in addition to the already assumed inequalities $\omega_{be} \gg \omega, \nu_e$, the turbulence correlation frequency ω is much higher than the collision frequency ν_e , we can average the kinetic equation (11) over the turbulent fluctuations. Since the turbulent drift (6) is incompressible in the (ψ, α_ψ) plane, according to Sec. II, this averaging amounts to introducing a pure diffusion in the ψ direction. Then we obtain the reduced kinetic equation,

$$\partial_t f(\mu, J, \psi, t) = \partial_\psi [D^{\psi\psi}(\mu, J, \psi) \partial_\psi f] + C(f), \quad (14)$$

in which all properties of turbulence are concentrated in the single turbulent diffusion coefficient $D^{\psi\psi}$, which is only different from zero in the trapped region $J < J_c(\mu, \psi)$. (See Ap-

pendix B for a more accurate estimate of $D^{\psi\psi}$ for passing electrons.) The opposite limit, $\omega_{be} \gg \nu_e \gg \omega$, is considered in Appendix A.

In addition to turbulence and collisions, particle and energy sources, when significant, must be included in the right-hand side of Eq. (14). Here we do not consider the effect of the sources.

It is emphasized that, while the diffusion over ψ was introduced on purely Hamiltonian grounds and on time scales much shorter than the collision time, the collision term in Eq. (14) can never be neglected, and, in a sense, it is the main term in the kinetic equation. On the other hand, the formally small turbulent diffusion term in (14) cannot be neglected either, because of the strong anisotropy of the phase space transport: collisions dominate the fluxes in the “velocity directions” of μ and J , whereas the turbulence determines the flux in the radial ψ direction, and this flux depends on the collisionally established distribution over μ and J .

If the collision operator is omitted, the formal steady-state solution to Eq. (14) is independent of ψ , meaning the constant per-unit-flux density $N_e(\psi) = \int f(\mu, J, \psi) d\mu dJ = \text{const.}$ ⁴² The usual Cartesian density in this turbulent equipartition is $n(\psi) \propto 1/\mathcal{V}'(\psi)$, where $\mathcal{V}'(\psi)$ is the volume inside the flux surface,

$$\mathcal{V}'(\psi) = \oint d\ell_{\parallel}/B \equiv \int_{-\pi}^{\pi} L(\psi, \theta) d\theta/B(\psi, \theta). \quad (15)$$

The “specific volume” (15) plays a prominent role in the MHD stability theory.⁵³

The restriction of the ψ -diffusion to the trapped region only and the predominance of collisions drive plasma to more complicated profiles. In the next section we derive electron transport equations by expanding in the small ψ -diffusion term in Eq. (14).

IV. ELECTRON TRANSPORT EQUATIONS IN TOROIDAL GEOMETRY

In this section we apply the Chapman-Enskog perturbation procedure to the kinetic equation (14) and derive the electron particle and energy fluxes driven by the given low-frequency turbulence.

As stated in Sec. III, the collisional relaxation occurs on the collision time scale τ_e ; that is, well before an electron is significantly displaced in the radial direction. It means that the distribution function $f(\mu, J, \psi)$ is locally Maxwellian, and the locality refers to flux surfaces. The local Maxwellian (13) contains two arbitrary functions of ψ : the density n and the temperature T . The slow evolution of these functions is determined by the turbulent diffusivity $D^{\psi\psi}$. Formally, one seeks solution of Eq. (14) by successive approximations in the small $D^{\psi\psi}$. The zeroth-order solution (13) follows from $C(f_0) = 0$. Since we neglect the neoclassical part, $\partial_{\psi}\Gamma^{\psi}$, of the bounce-average electron collision operator, it conserves particles and energy locally:

$$\int C(f) dJ d\mu = 0, \quad \int \epsilon(\mu, J, \psi) C(f) dJ d\mu = Q_{ie}, \quad (16)$$

where Q_{ie} is the usual, collisional energy exchange between ions and electrons. The first-order solution $f_1(\mu, J, \psi, t)$ satisfies

$$\partial_t f_0 = \partial_{\psi} [D^{\psi\psi}(\mu, J, \psi) \partial_{\psi} f_0] + C(f_0 + f_1). \quad (17)$$

The actual perturbation f_1 need not be calculated; for the evolution of $n(\psi, t)$ and $T(\psi, t)$ we only need two solubility conditions obtained from Eq. (17) by integrating it over μ and J with the weights of 1 and ϵ , respectively. Using the conservation laws (16), this results in the electron particle and energy balance equations:

$$\partial_t N_e = -\partial_{\psi} \Gamma_e, \quad (18)$$

$$\partial_t W_e = -\partial_{\psi} Q_e + Q_{ie} + Q_{ie}. \quad (19)$$

Here, in order to emphasize conservation laws, we use the per-unit-flux electron density $N_e(\psi, t) = n(\psi, t) \mathcal{V}'(\psi)$ and the analogously defined electron energy density $W_e = (3/2) N_e T$. The dimensions of Γ_e , q_e , and Q_{ie} are particles/s, ergs/s, and ergs/(Wb s), respectively.

Equations (18) and (19) involve the following particle and energy fluxes through the whole magnetic surface:

$$\Gamma_e = - \int d\mu dJ D^{\psi\psi} \partial_{\psi} f_0, \quad q_e = - \int d\mu dJ D^{\psi\psi} \epsilon \partial_{\psi} f_0. \quad (20)$$

In addition to the classical ion-electron energy exchange Q_{ie} , Eq. (19) contains another source term,

$$Q_{ie} = - \int d\mu dJ (\partial_{\psi} \epsilon_K) D^{\psi\psi} \partial_{\psi} f_0, \quad \epsilon_K \equiv \epsilon - e\phi = \mu \mathcal{B}(\psi, j), \quad (21)$$

interpreted as the energy exchange between the turbulence and the electrons. It is due to the inevitable variation of the particle kinetic energy $\epsilon_K(\mu, J, \psi)$ in course of displacement in ψ under conserved μ and J . The flux q_e and the exchange term Q_{ie} are defined up to a gauge leaving $-\partial_{\psi} q_e + Q_{ie}$ unchanged. Equation (21) uses the gauge in which Q_{ie} vanishes in a uniform magnetic field.

Substitution of Eq. (13) into (20) and (21) yields the desired expressions for the fluxes and the energy exchange. These expressions can be made more explicit by writing the diffusivity $D^{\psi\psi}$ as a function of μ , j , and ψ and Taylor expanding it in μ :

$$D^{\psi\psi}(\mu, j, \psi) = \sum_{l=0}^{\infty} D_l(j, \psi) \mu^l / l!. \quad (22)$$

Then we can write $d\mu dJ = (\mu m)^{1/2} d\mu dj$ and the integrals over μ are solved explicitly. This gives the following:

$$\begin{pmatrix} \Gamma_e \\ q_e \\ Q_{ie} \end{pmatrix} = - \sum_{l=0}^{\infty} \frac{(2l+1)!!}{\sqrt{8}(2l)!!} \int_0^{\infty} \frac{D_l dj}{\mathcal{B}^{3/2}} \left(\frac{T}{\mathcal{B}} \right)^l \begin{pmatrix} 2n \\ nT \\ nT \mathcal{B}' / \mathcal{B} \end{pmatrix} \times \left[\frac{n'}{n} + \begin{pmatrix} l \\ l+1 \end{pmatrix} \frac{T'}{T} - \begin{pmatrix} l+3/2 \\ l+5/2 \end{pmatrix} \frac{\mathcal{B}'}{\mathcal{B}} \right], \quad (23)$$

where the prime denotes differentiation with respect to ψ . We see that the fluxes Γ_e and q_e contain the usual terms with the density and the temperature gradients and also the pinch term with $\partial_{\psi}\mathcal{B}(j, \psi)$ arising from the inhomogeneity of the magnetic field.

The expansion (22) is motivated by the possibly weak dependence of $D^{\psi\psi}$ on μ at the fixed pitch angle j and radius ψ . The reason is that j defines the extent of the particle bounce motion, whereas μ determines how fast this motion occurs. This does not affect the bounce-average fluctuation Hamiltonian $H_1(j, \psi, \alpha_{\psi}, t)$, but μ enters the unperturbed Hamiltonian $H_0 = e\phi_0(\psi) + \mu\mathcal{B}(j, \psi)$ (where ϕ_0 is the potential of the radial electric field) and affects the rate of the toroidal precession $\omega_{\varphi} = -(2\pi c/e)\partial_{\psi}H_0 = qc(-ed\phi_0/dr + \mu B/R)/(eBr)$. The toroidal drift resonance condition $\omega = k_{\varphi}R\omega_{\varphi}$ then leads to a μ -dependent turbulent diffusivity $D^{\psi\psi}$ in as much as the quasi-linear theory⁴⁰ applies and the μ -dependence of the toroidal precession is significant. As the plasma potential ϕ_0 is typically of order the temperature, the dependence of ω_{φ} on μ is proportional to r/R and, for a thin torus, can be neglected.

Thus, depending on the properties of turbulence and, in particular, in a thin torus, the dependence of $D^{\psi\psi}(\mu, j, \psi)$ on μ can be weak, and in the series (22) only the D_0 term may be retained. In this case, Eqs. (23) are simplified as follows:

$$\Gamma_e = -\hat{D} \frac{dn}{d\psi} + \hat{V}n, \quad (24)$$

$$q_e = -\frac{3}{2}\hat{D} \frac{d(nT)}{d\psi} + \frac{5}{2}\hat{V}nT, \quad (25)$$

$$Q_{te} = -\hat{V} \frac{d(nT)}{d\psi} + \hat{U}nT, \quad (26)$$

where

$$\hat{D}(\psi) = (1/\sqrt{8}) \int dj D_0(j, \psi) \mathcal{B}^{-3/2}(j, \psi), \quad (27)$$

$$\hat{V}(\psi) = (3/2\sqrt{8}) \int dj D_0 \mathcal{B}^{-5/2} \partial_{\psi} \mathcal{B}, \quad (28)$$

$$\hat{U}(\psi) = (15/4\sqrt{8}) \int dj D_0 \mathcal{B}^{-7/2} (\partial_{\psi} \mathcal{B})^2. \quad (29)$$

The integration in (27)–(29) extends over the trapped region $0 < j < j_c$, where the diffusivity $D_0 \neq 0$.

In the approximation of the weak dependence of the diffusivity $D^{\psi\psi}(\mu, j, \psi)$ on the trapped particle energy (via μ), the particle flux (24) has no cross term with the temperature gradient, because the toroidal drift resonance is primarily determined by the radial electric field. The cross term appears in higher orders of the expansion (23). In a thin torus, $\hat{V} \propto r/R$ and $\hat{U} \propto (r/R)^2$. As shown in Sec. V, the pinch velocity \hat{V} is usually negative (inward).

The energy exchange term (26) in Eq. (19) appears as a local source term. If, however, we wish to include it into the divergence of the heat flux q_e , the flux will change by the non-locally defined quantity $\delta q_e = -\int_0^{\psi} Q_{te} d\psi$. Thus the turbulence-particle energy exchange results in certain nonlocal features. This kind of nonlocality alone is insufficient to

explain the ‘‘cold pulse’’ experiment⁷ in which a negative electron temperature perturbation reverses sign during inward propagation. The mechanism of this effect must depend on the nondiffusive processes *and* a nonlinear change in the turbulent transport coefficients.

The energy of the turbulent fields themselves is negligible in comparison with the thermal energy of particles, and if electrons release some energy to the turbulence at some radius ψ , the same energy is returned by the turbulence to particles (electrons or ions) at the same or some other radius ψ . Although wave energy transfer mechanisms are possible,⁵⁴ the predominantly negative, for the decreasing electron pressure $dp_e/dr < 0$, value of Q_{te} implies a local anomalous electron-ion energy exchange. This turbulence-enhanced electron-ion energy exchange channel has not been previously taken into account in power-balance analyses. This electron energy loss mechanism could affect the nondimensional transport scalings⁵⁵ inferred from the power balance with a classical energy exchange. The sign of the effect also helps to alleviate the existing difficulties with the explanation of the electron energy losses by fluctuation transport.⁵⁶ Normalizing to unit volume and neglecting the second term in Eq. (26), the electron energy exchange power $Q_e = Q_{te}/\mathcal{V}'(\psi)$ can be written

$$Q_e = -V_e \frac{dp_e}{dr}, \quad (30)$$

where $V_e < 0$ is the measured in m/s local electron pinch velocity, and a negative Q_e means energy transfer from the electrons to the ions.

V. CANONICAL PROFILES

In this section we investigate the density and temperature profiles in the absence of particle and energy fluxes. These relaxed profiles play the role of canonical profiles in the sense that the tokamak plasma tends to relax to these profiles when the particle and energy sources are insignificant in comparison with the turbulent transport.

The steady-state solution to Eqs. (18) and (19), neglecting the collisional energy exchange Q_{ie} , defines the relaxed profiles of the density $n_0(\psi)$ and the electron pressure $p_0(\psi) = n_0(\psi)T_0(\psi)$ as follows (prime denotes $d/d\psi$):

$$\begin{aligned} (\log n_0)' &= \hat{V}(\psi)/\hat{D}(\psi), \\ \frac{3}{2}(\hat{D}p_0)' - \frac{1}{2}\hat{V}p_0' + (\hat{U} - \frac{5}{2}\hat{V}')p_0 &= 0. \end{aligned} \quad (31)$$

Equations (31), as (24)–(26), relying on the zeroth-order expansion (22), are only valid for a thin torus with $r/R \ll 1$ and can be further simplified. The transport coefficients \hat{D} , \hat{V} , and \hat{U} are determined by trapped particles only. In the trapped region, $j < j_c$, we have $B_{\min}(\psi) < \mathcal{B}(j, \psi) < B_{\max}(\psi)$, and thus the derivative $\partial_{\psi} \mathcal{B} \sim (r/R)B/\psi$ is small. So is the pinch term in (24), and the relaxed density profile in this approximation is only weakly inhomogeneous:

$$n_0(\psi) = n_0(0) \left[1 + \int_0^{\psi} d\psi \frac{\hat{V}(\psi)}{\hat{D}(\psi)} + O\left(\frac{r^2}{R^2}\right) \right]. \quad (32)$$

Here the integral term is $O(r/R)$. Similarly, the second equation (31) yields

$$p_0(\psi) = p_0(0) \left[1 + \frac{5}{3} \int_0^\psi d\psi \frac{\hat{V}(\psi)}{D(\psi)} + O\left(\frac{r^2}{R^2}\right) \right]. \quad (33)$$

The resulting ‘‘adiabatic’’ profile $p_0(\psi) \propto n_0^{5/3}(\psi)$ is the property of the expansion to $O(r/R)$; it does not follow from the underlying kinetic equation (14).

In general, the relaxed profiles of n_0 and p_0 depend on both the magnetic geometry and the distribution of turbulence via $D_0(j, \psi)$. To obtain more explicit canonical profiles, we make another, uncontrollable approximation³³ whereby $D_0(j, \psi) = \Theta[j_c(\psi) - j]D_0(\psi)$; that is, the ψ diffusion of trapped electrons is independent of the trapping depth. This would be the case if the parallel turbulence correlation length were much larger than the connection length L (the standard assumption is that the two length are of the same order). Then the arbitrary amplitude $D_0(\psi)$ cancels in Eq. (32), and the result depends on the magnetic geometry only. In the case of circular magnetic surfaces the result is (see Appendix A):

$$n_0(r) = n_0(0) \left[1 - \frac{4}{3R} \int_0^r dr \left(\frac{d \log q}{d \log r} + \frac{3}{8} \right) + O\left(\frac{r^2}{R^2}\right) \right]. \quad (34)$$

The corresponding pressure profile is similar and involves the coefficient of 20/9 in place of the 4/3 in Eq. (34).

VI. DISCUSSION

Our analysis was focused primarily on electrons for which the effect of typical turbulent fluctuations was identified in terms of the trapped electron diffusion over the poloidal flux coordinate ψ . We stressed that it does matter in which coordinate the pure diffusion of particles takes place: the nonlinear relation between ψ (in which the bounce-average turbulent diffusion is pure) and the radius r means an average radial pinch velocity. Thus the observed convective fluxes in tokamaks have firm theoretical grounds in terms of the conservation of adiabatic invariants during the chaotic particle motion and the Liouville theorem constraints acting on turbulent time scales, so far as collisions are ignored. Taking collisions into account is possible in a regular fashion, and the result is a set of transport equations, (18) and (19), involving anomalous, both diffusive and convective, particle and energy fluxes, and an anomalous electron-ion energy exchange [Eqs. (23) or (24)–(26)].

The transport coefficients depend on the parameters of turbulence, which is self-consistently related to plasma fueling and profiles; however, the structure of the electron transport equations is important in its own right even when the turbulence part of the theory is missing. A notable feature of this structure is the absence of any conspicuous Onsager symmetry.^{57,58} Even though the distribution function is very close to Maxwellian, the usual entropy-based arguments behind the Onsager symmetry fail because of the predominantly turbulent mechanisms of the plasma profiles relax-

ation. The pinch effect alone renders the usual transport matrix nonexistent⁵⁹ or at least prompts including new ‘‘thermodynamic forces’’ such as ∇B .

Although ions do not conserve their longitudinal invariant J , it does not mean the absence of the anomalous pinch effect for the ions: The conservation of μ alone induces an invariant measure with the density proportional to the magnetic field. In a thin torus, this is an $O(r/R)$ effect, just as the collisional pinch effect for electrons. The ion pinch effect can be formally described by an action-diffusion equation in which only μ -components of the diffusion tensor are zero. In any event, the effects of the magnetic geometry on local transport and the profile formation have to be taken into account in some form. An interesting problem regarding specifically ions is the momentum transport, because of the strong back reaction of plasma rotation on turbulence.^{60,61} There is experimental evidence of the pinch effect for the momentum transport.⁶²

It is always necessary to remember the limitations intrinsic to the test-particle approach, which we used so far. By prescribing turbulence in terms of action diffusion coefficients, for example, we can easily run into the trouble of violating the quasi-neutrality of plasma with few impurities. This could not be a problem for energy transport, but, in the analysis of the particle transport, the quasi-neutrality constraint should be built into microscopic field equations long before any averaging and diffusion approximation are endeavored. Of course, each plasma particle, electron or ion, does move in the turbulent fields, whatever they are, and the resulting diffusion or diffusion-convection description is equally appropriate for both electrons and ions. On the level of averaged description, it is the relation between the various transport coefficients, which enforces (or reflects) self-consistency and, particularly, quasi-neutrality.

In this connection, it is appropriate to ask as to which plasma species, if any, is more amenable to the test-particle analysis. The simple answer is that both electrons and ions are correctly described by averaged transport equations, such as (4), *correctly derived* from the underlying single-particle Hamiltonian dynamics and supplied with an appropriate collision operator. The more complex answer is that one never knows the properties of turbulence from pure theory and therefore makes various approximations to obtain estimates of the transport coefficients. The applicability of the test-particle approach must be then defined as the reliability of simple decorrelation-based estimates, such as $D^{\psi\psi} \sim \langle \dot{\psi}^2 \rangle / \omega$, and the like.

In our opinion, electrons are better suited for the test-particle analysis for the following reasons. First and most of all, the fast parallel motion and the associated adiabatic invariants μ and J make the electron dynamics, while chaotic and complex, effectively less multidimensional and simpler than the ion dynamics. The bounce-average drift of trapped electrons is two dimensional (ψ, α_ψ) and the transport-relevant Fokker-Planck description is one dimensional (ψ). The corresponding dimensions for the ions are four and two. On the one hand, the parallel ion motion is strongly coupled to the cross-field transport and, on the other, the existence of the whole phase-space diffusion matrix in the (ψ, J) plane

makes it harder to predict the relaxed density and temperature profiles, given the unavoidable uncertainties in the diffusion matrix components. The significant banana width of the ion orbits introduces further complications. The lesser arbitrariness in the electron transport equations allows us to derive plasma density profiles based on electrons alone.

The usual reproach cast upon the test-particle analysis is that it predicts different turbulent diffusivities for the ions and the electrons. This is also true in the toroidal geometry. Recalculated for the minor radius r , the quasi-linear estimate of the ion diffusivity is $D_i \sim (c\tilde{E}_\theta/B_\varphi)^2/\omega$, whereas the electron diffusivity, $D_e \sim (r/R)^{1/2}(c\tilde{E}_\varphi/B_\theta)^2/\omega$, is proportional to the fraction $(r/R)^{1/2}$ of trapped particles subject to the average drift (6), or $\dot{r} = -c\tilde{E}_\varphi/B_\theta$. As the parallel electric field $\tilde{E}_\parallel = (\tilde{E}_\theta B_\theta + \tilde{E}_\varphi B_\varphi)/B$ is usually small, we have $\tilde{E}_\theta/B_\varphi = -\tilde{E}_\varphi/B_\theta$, and so $D_i/D_e \sim (R/r)^{1/2}$ turns out large in a thin torus. One possibility is that the intrinsic correlations between the parallel ion motion and the turbulent fluctuations make the ion diffusivity much smaller than the simple quasilinear estimate would predict. (This is the only option for the case of a uniform magnetic field.) Another possibility of enforcing the quasi-neutrality lies in the pinch effect. The fact that $D_i \gg D_e$ can simply mean that the diffusive and the convective fluxes of the ions are in a fine balance yielding total particle flux much less than $D_i dn/dr$. This possibility also raises implications for the calculation of the radial electric field other than usual.⁶³

This example, as all the preceding analysis, emphasizes the role of convective—that is, independent of the density and temperature gradients—fluxes in toroidal geometry. These fluxes, or the pinch effect, shape the “canonical” profiles. These profiles depend on the properties of turbulence (itself driven by the gradients) but, under certain approximations, are determined by the magnetic geometry only. Upon accepting the inhomogeneous canonical profiles as a new frame of reference, the particle and heat diffusivities are no longer the measure of losses but rather the measure of the resilience of the canonical profiles to externally imposed fluxes. Under this paradigm, the routinely used transport coefficients, such as χ_i , are no longer crucial confinement parameters. For example, one can safely set $\chi_i = \infty$ to merely say that the ion temperature keeps its canonical profile, no matter what plasma heating and energy flux there are. In this extreme limit, the crucial problem of determining the central temperature boils down to determining the canonical T_i profile and attaching this profile to an appropriate boundary condition at the plasma edge,²⁸ which highlights the role of the plasma edge physics in the global tokamak confinement.

Both the energy pinch and the anomalous energy exchange are important players in the power balance, and they must be included in studies involving local power-balance analysis, such as the nondimensional ρ_* -scaling studies.^{55,64} The specific predictions regarding the relation between the diffusivity, the pinch velocity, and the exchange rate, such as those in Eqs. (24)–(26) and (27)–(29), could be useful in such analyses.

ACKNOWLEDGMENTS

M.I. wishes to thank V. V. Yankov and A. H. Boozer for useful critical remarks and A. J. Wootton and L. Chen for helpful discussions. P.D. would like to acknowledge the critical discussions with W. M. Tang.

This work was supported by the U.S. Department of Energy under Grants No. DE-FG05-80ET53266 and No. DE-FG03-94ER-54241.

APPENDIX A: LARGE-ASPECT-RATIO LIMIT

In the limit of a thin torus with circular flux surfaces, $r \ll R \approx \text{const}$, we have

$$B = B_0 \left(1 - \frac{r \cos \theta}{R} \right), \quad q(r) = \frac{r B_0}{R B_\theta(r)}, \quad (A1)$$

$$\psi(r) = 2\pi B_0 \int_0^r \frac{r dr}{q(r)}, \quad L(r) = q(r)R,$$

and θ is the geometrical poloidal angle around the magnetic axis. Then the pitch-angle variable (9) is

$$j = j_c I(z), \quad j_c = 8q(B_0 R r)^{1/2}, \quad z = \frac{1}{2} + \frac{R}{r} \frac{\mathcal{B} - B_0}{2B_0}, \quad (A2)$$

where the function I is expressed through the complete elliptic integrals E and K :

$$I(z) = \begin{cases} E(z) - (1-z)K(z), & 0 < z < 1 \text{ (trapped)}, \\ \sqrt{z}E(1/z), & z > 1 \text{ (passing)}. \end{cases} \quad (A3)$$

Solving Eq. (A2) for $\mathcal{B}(j, \psi)$ yields: $\mathcal{B} = B_0 + B_0(r/R) \times [2z(j/j_c) - 1]$, where the function $z(I)$ is the inverse to (A3). In the trapped region $j < j_c$ we have $z(I) < 1$ and thus $\mathcal{B}(j, \psi)$ is a weak function of radius:

$$\frac{\partial \mathcal{B}(j, r)}{\partial r} = \frac{B_0}{R} \left[2z(I) - 1 - 2Iz'(I) \left(\frac{d \log q}{d \log r} + \frac{1}{2} \right) \right],$$

$$I = \frac{j}{j_c(r)}. \quad (A4)$$

Upon substituting Eq. (A4) into (28) and (32), result (34) follows.

The $O(r/R)$ pinch effect obtained here for collisionless electrostatic modes with $\omega_{be} \gg \omega \gg \nu_e$ remains also small for very-low-frequency modes with $\omega_{be} \gg \nu_e \gg \omega$. In this limit, the collisionless averaging over fluctuations cannot be done, and one has to start with Eq. (11) instead of (14). The same perturbation expansion, $f = f_0 + f_1 + \dots$, can be used in this case; however, in order to infer the fluxes, one has to actually solve the first-order equation,

$$(2\pi c/e)[H_1(\mu, J, \psi, \alpha_\psi, t), f_0] + C_1(f_1) = 0, \quad (A5)$$

describing the quasi-static collisional response of electrons, f_1 , to the toroidally asymmetric potential perturbation H_1 . This amounts to the inversion of the linearized collision operator $C_1(f_1)$:

$$f_1(\mu, J, \psi, \alpha_\psi, t) = (2\pi c/e) C_1^{-1} \times [\partial_{\alpha_\psi} H_1(\mu, J, \psi, \alpha_\psi, t) \partial_\psi f_0(\mu, J, \psi)]. \quad (\text{A6})$$

The Maxwellian f_0 depends on the magnetic geometry only via $\mathcal{B}(j, \psi)$, a weak function of ψ in the trapped region, so the pinch term in the flux due to the perturbation f_1 is again $O(r/R)$. The cross fluxes [Γ due to $T'(\psi)$ and q_e due to $n'(\psi)$] are generally not small.

APPENDIX B: TRANSPORT OF PASSING ELECTRONS

Each adiabatic invariant $I = \oint [m\mathbf{v} + (e/c)\mathbf{A}] \cdot d\mathbf{l}$ involves the magnetic flux through the relevant nearly closed particle orbit. A natural extension of the adiabatic invariance to passing particles can be attempted for a rational magnetic surface, $q(\psi) \equiv d\chi/d\psi = m/n$, where the magnetic field line is closed upon n poloidal and m toroidal revolutions: $I(\psi) = (e/c)[n\chi(\psi) - m\psi]$. The conservation of $I(\psi)$ seems to imply $\dot{\psi} = \text{const}$, that is, no turbulent transport of passing electrons.²⁹ In this simple form, however, this argument is not definitive, because the change of I in response to an attempted radial displacement is small: $dI/d\psi = (e/c) \times (nq - m) = 0$, by our definition of q . We thus expect that the radial transport of passing particles, even for low-frequency fluctuations, can be finite, and an estimate of the effect is needed.

The transport will arise when the resonance condition $\omega = k_{\parallel} v_{\parallel}$ is satisfied for a non-exponentially weak part of the fluctuation spectrum. Here the parallel wavenumber $k_{\parallel} = \mathbf{k} \cdot \mathbf{b} = (m+nq)/L$. The standard assumption that $k_{\parallel} L \sim 1$ implies the fluctuation spectrum lying within the strip $|m+nq| \lesssim 1$ on the (m, n) plane. The resonance takes place for the subset of modes with $|m+nq| \lesssim \omega/\omega_{be} \sim 1/10$. As the typical fluctuation harmonics in tokamaks are high, $|m| \sim |n|q \sim 30-50$, the resonant modes are not improbable. For an irrational q and large n , the quantity $\Delta_n = \min_m |m+nq|$, the distance from nq to the nearest integer, is a version of the standard computer random-number generator with the uniform distribution in $[0, 1/2]$. Given $n \gg 1$ trials, the minimum Δ_n is of order $1/n$, which is less than ω/ω_{be} . Thus there do exist a few (m, n) fluctuation modes, which are resonant with passing electrons and induce their transport with finite diffusivities $D^{\psi\psi}$, $D^{\psi J}$, and D^{JJ} .

The magnitude of the diffusivity $D^{\psi\psi}$ is given by the integral of the $\dot{\psi}$ correlator:

$$D^{\psi\psi} = \int_0^\infty C^{\psi\psi}(t) dt, \quad C^{\psi\psi}(t) = \langle \dot{\psi}(0) \dot{\psi}(t) \rangle. \quad (\text{B1})$$

Expanding the potential $\phi(\psi, \theta, \varphi, t) = \phi(\psi, \theta, \alpha_\psi + q\theta, t)$ in a Fourier series over the angles θ and φ and in a Fourier integral over time, and assuming random phases as in the quasi-linear theory, $\langle \phi_{mn\omega} \phi_{m'n'\omega'} \rangle = I_{mn\omega} \delta_{m+m'} \delta_{n+n'} \times \delta(\omega + \omega')$, we find from $\dot{\psi} = 2\pi c \partial_\varphi \phi$:

$$C^{\psi\psi}(t) = (2\pi c)^2 \sum_{mn\omega} n^2 I_{mn\omega} e^{i(m+nq)\theta(t) - i\omega t}. \quad (\text{B2})$$

For the model spectrum $I_{mn\omega} = (2\pi)^{-1/2} I_{mn} \times \omega_{mn}^{-1} \exp(-\omega^2/2\omega_{mn}^2)$, Eq. (B2) takes the form

$$C^{\psi\psi}(t) = (2\pi c)^2 \sum_{mn} n^2 I_{mn} e^{i(m+nq)\theta(t) - \omega_{mn}^2 t^2/2}. \quad (\text{B3})$$

For passing particles, $\theta(t)$ is the ballistic motion $\omega_{be} t$ plus a periodic oscillation due to the velocity change in the magnetic well; the simplest model is $\theta(t) = \omega_{be} t + \delta_1 \sin(\omega_{be} t)$. Then, expanding $e^{i(m+nq)\delta_1 \sin(\omega_{be} t)} \approx 1 + i(m+nq)\delta_1 \sin(\omega_{be} t)$, we calculate the diffusivity (B1):

$$D^{\psi\psi} = \sqrt{2\pi} (2\pi c)^2 \sum_{mn} n^2 I_{mn} \left[e^{-(m+nq)^2/2\nu_{mn}^2} + \frac{(m+nq)\delta_1}{2} (e^{-(m+nq+1)^2/2\nu_{mn}^2} - e^{-(m+nq-1)^2/2\nu_{mn}^2}) \right]. \quad (\text{B4})$$

Here the small parameter $\nu_{mn} = \omega_{mn}/\omega_{be} \ll 1$ is responsible for selecting the resonant modes with $|m+nq \pm l| < \nu_{mn}$, where the definition of the resonant modes is generalized to account for the periodic change in the particle parallel velocity. For the purpose of estimate, the $l \neq 0$ terms in (B4) can be ignored. The number of (m, n) modes selected by the requirement of the parallel resonance is determined by the probability for nq to be within a ν_{mn} -vicinity of an integer. As discussed above, the probability is of order $\nu_{mn} \ll 1$. Thus the sum (44) is estimated as

$$D^{\psi\psi} \approx \sqrt{2\pi} (2\pi c)^2 \sum_{mn} n^2 I_{mn} \nu_{mn} \approx (2\pi c R \tilde{E}_\varphi)^2 / \omega_{be}. \quad (\text{B5})$$

A similar estimate for trapped particles involves the turbulence correlation time ω in place of ω_{be} .

Thus the transport of passing electrons can be neglected in comparison with trapped electrons if $\omega/\omega_{be} \ll (r/R)^{1/2}$. In this case, the particles get collisionally trapped and undergo a faster ψ -diffusion well before the slow passing transport accumulates an appreciable effect.

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